GENERAL GEOMETRIC LÉVY PROCESSES FOR ASSET PRICES MODELLING

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Abstract

In this study , the stock prices process is modelled by a stochastic differential equation driven by a general Lévy process. We review some fundamental mathematics properties of Lévy distribution. Except for the Geometric Brownian Model and the Geometric Poissonian Model , General Lévy Market Models are incomplete models and there are many equivalent martingale measures. We show that, using the Power – Jump Processes the market can be complated. Furthermore we give sufficient conditions for no arbitrage in Lévy market and in completeness of a Lévy Market, and finally we show that ,the pricing formula for contingent claims of European type and the problem of a choice of equivalent martingale measures.

Keywords :Lévy Processes, Market Models, Martingales, Arbitrage, Option Pricing.

1. INTRODUCTION

Normality assumption of asset returns has played a central role in financial theory starting with the Markowitz frontier and Capital Asset Pricing Model (CAPM) and Black-Scholes model. The normality of distributions has been augmented with the assumption of continuity of trajectories when Samuelson introduced in 1965 the Geometric Brownian motion, then used in the seminal papers by [4], [8].

In the basic Black-Scholes model, the price of a stock (or index) follows the Geometric Brownian motion $S_t = \exp X_t$, where X_t is Brownian motion. The probability density of the increments $X_{t+\Delta t} - X_t$ decay faster than an exponential function as $x \to \pm\infty$ .Return distributions are more leptokurtic than the normal one as noted by Fama as early as 1963 ; this feature is more accentuated when the holding period becomes shorter and becomes particularly clear on high frequency data. From the begining of the 90 th, several families of Levy processes with probability densities having semi-heavy, that is exponentialy decaying tails heve been used to model stock returns:Variance Gamma processes, [7] ,Normal Inverse Gaussian Processes,[1],Generalized Hyperbolic Processes[6].

Processes of all the families above heve been show to fit better to the dynamics of historic prices and pricing formulas for European options based on these processes, also perform better than earlier models. Until recently , almost no effective analytical formulas in pricing of the European options. The main goal of the paper is to partially fill in this gap.We will work under a so-called Geometric Lévy Market Model. Under this model the stock price process $S = \{S_t, t \geq 0\}$ is modelled by a Stochastic Differential Equation (SDE) driven by a general Lévy process $Z = \{Z_t, t \geq 0\}$,

$$\frac{dS_t}{S_t} = b\,dt + dZ_t \quad , \qquad S_0 > 0 . \tag{1}$$

Classical Black-Scholes model is taking for the Lévy process Z a Brownian motion . Black-Scholes model is a so-called *complete* model, in that all contingent claim can be duplicated by a portfolio consisting of investments in the stock and in a risk free bond: The risk of any claim can be completely hedged against. In such a complete model there exists a unique equivalent martingale measure and the unique price of a contingent claim is just the discounted expectation of the payoff at maturity.

Except for the geometric Brownian model and and geometric Poissonian model, for the above described general Lévy market models , there are many equivalent martingale measures and such markets are incomplete : Contingent claims cannot in general be hedged by a portfolio.

The paper is organized as follows.In section 2,We recall some basic resuls on Lévy Processes, in section 3, we take as given an equivalent martingale measure then we show that the market is complete , in section 4,we look whether there exists an equivalent martingale measure making all the discounted traded assets martingales, in section 5,we present some regards.

2. THE GEOMETRIC LÉVY MODEL

In this section , we shortly recall some basic results on Lévy processes,for details ,you can look [3],[10],[11].

2.1 Lévy Processes

We represent the uncertainity of the economy by a filtered probability space $(\Omega, \Im, (\Im_t)_t, P)$ where Ω can view as a set all trajectories of the process and $\Im_t$ is the filtration of information available at time t and P is the real probability measure.

A process $Z = \{Z_t, t \geq 0\}$ is called a *Lévy Process* if,

i) For every $s,t \geq 0$, $Z_{t+s} - Z_s$ is independent of Z_s .(independent increments).

ii) $Z_0 = 0$, P- a.s.

iii) The distribution of $Z_{t+s} - Z_s$ does not depend of s.(Temporal homogeneity or stationary increments property)

iv) It is stochastically continuous, i.e.

$$\lim_{t \to 0^+} P\left[|Z_t| > \varepsilon\right] = 0 \quad \textit{for any} \quad \varepsilon > 0$$

Since any process Z_t satisfying (i)-(iv) has a cadlag modification, we will assume Z_t to be cadlag.

$\psi(z) = \log \phi(z) = \log E[\exp(izZ_1)]$ is called the *characteristic exponent* and it satisfies the following *Levy- Khintchine formula*(this formula describes explicitly a Levy process in terms of its Fourier transform) :

$$\psi(z) = i\alpha z - \frac{c^2}{2}z^2 + \int_{-\infty}^{+\infty}\left(e^{izx} - 1 - izx 1_{\{|x|<1\}}\right)v(dx)$$

where $\alpha \in \Re$, c > 0, and v is a measure on $\Re / \{0\}$ with $\int_{-\infty}^{+\infty}\min(1, x^2)v(dx) < \infty$. Our infinitely divisible distribution has a triplet of Levy characteristics $[\alpha, c^2, v(dx)]$. The measure $v(dx)$ is called the Levy measure of Z , $v(dx)$ dictates how the jumps occur. If v=0 the process is gaussian and if $c^2 = 0$, the Levy process is a pure non-Gaussian process without the diffusion component.

From the Levy-Khintchine formula, Z must be a linear combination of a standart Brownian motion, $W = \{W_t ; t \geq 0\}$ and a pure Jump process, $X = \{X_t ; t \geq 0\}$:

$$Z_t = cW_t + X_t$$

where W is independent of X . Moreover, we will suppose that the Levy measure satisfies,

$$\int_{-\infty}^{+\infty}|x|^i v(dx) < \infty \quad , \quad i \geq 2$$

and characteristic function $E[\exp(iuX_t)]$, so all moments of Z_t $(and\ X_t)$. Note that,

$$\alpha = E[X_1] - \int_{|x| \geq 1} x v(dx)$$

The Doob decompozation of X, in terms of a martingale part and a predictable process of finite variation is given by

$$X_t = L_t + at$$

where $L = \{L_t ; t \geq 0\}$ is a martingale and $E[X_1] = a$

2.2 Power Jump Processes

We define the following transformation of $Z = \{Z_t \, ; t \geq 0\}$,

$$Z_t^i = \sum_{0 < s \leq t} (\Delta Z_s)^i \quad , \quad i \geq 2$$

where $\Delta Z_s = Z_s - Z_{s^-}$. Let $X_t^i = Z_t^i$, $i \geq 2$. The process $X^{(i)} = \{X_t^i \, ; t \geq 0\}$, i=1,2,3,... is again a Levy process and is called the *Power –Jump process of order i.*They jump at the same points as the original Levy process but the jumps sizes are the power of the jump size are the i th power of the jump size original Levy process.

We heve $E[X_t] = E[X_t^1] = ta = tm_1 < \infty$ (see [9],p.29) and

$$E[X_t^i] = t \int_{-\infty}^{+\infty} x^i v(dx) = m_i t < \infty \quad , \quad i \geq 2$$

We denote by

$$Y_t^i = Z_t^i - m_i t \quad , \quad i = 1, 2, 3, \ldots$$

the compensated ith- power jump process. $Y^{(i)} = \{Y_t^i \, ; t \geq 0\}$ is a normal martingale and was called the *Teugels martingale of order i*.

2.3 The Geometric Lévy Model

Using Ito formula for cadlag semimartingales one can show that,

$$\frac{dS_t}{S_{t^-}} = b \, dt + dZ_t \quad , \quad S_0 > 0$$

equation has an explicit solution;

$$S_t = S_0 \exp\left(cW_t + L_t + \left(a + b - \frac{c^2}{2} \right)t \right) \prod_{0 < s \leq t} (1 + \Delta L_s) \exp(-\Delta L_s).$$

In order to ensure that all $S_0 \geq 0$ for all $t \geq 0$ almost surely, we need $\Delta L_t \geq -1$ for all t. Levy measure v is supported on a subset of $[-1, +\infty]$. The riskless rate of interest we assume to be a constant r. The value of the riskfree bond or bank account at time t is then given by $B_t = \exp(rt)$.

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3. LÉVY MARKET MODEL

Suppose,we heve on equivalent martingale measure Q under which Z remains a Lévy process.Under this measure, the discounted stock price process is a martingale and the process , $\widetilde{Z} = \{Z_t + (b - r)t \ ; \ t \geq 0\}$ will be a Lévy process, moreever the process $\widetilde{Z}$ is a martingale.

We show how to calculate explicitly the hedging portfolio of a contingent claim of which the payoff is only a function of the value at maturity of the stock price i.e., $X = F(T, S_T)$.

The value of the contingent claim at time t is given by,

$$F(t, S_t) = \exp(-r(T-t)) E_Q[X|F_t]$$

We call $F(t, x)$ the price function of X. Denote by D_1 the differential operations with respect to the time variable and by D_2,

Differential operator with respec to the second variable (stock prices). Finaly, denote by D following integral operator :

$$DF(t, x) = \int_{-\infty}^{+\infty} (F(t, x(1+y)) - F(t, x) - xy\, D_2\, F(t, x))\, \widetilde{v}(dy)$$

The price function (at time t) $F(t, x)$ satisfies ;

$$D_1 F(t, x) + rx\, D_2 F(t, x) + \frac{1}{2} c^2 x^2 D_2^2 F(t, x) + DF(t, x) = rF(t, x),$$

3.1 Black- Scholes Model

Suppose our risk – neutral dynamics of the stock price are given by Black- scholes SDE,

$$\frac{dS_t}{S_t} = \left(r - \frac{1}{2}\sigma^2\right) dt + dW_t \quad , \quad S_0 > 0$$

where $W = \{W_t, t \geq 0\}$ is standard Brownian motion and stock price process, $S = \{S_t, t \geq 0\}$ is given by ,

$$S_t = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)$$

Market complate and hedging portfolio is given by $\dfrac{F(s, S_s) - S_s D_2 F(s, S_s)}{B_s}$ number

of bonds and $D_2 F(s, S_s)$ number of stocks.

3.1.1 Pricing Formulas

Consider the value at time t of a contingent claim X with a payoff function $f(S_T) = F(T, S_T)$ only depending on the stock price at maturity :

$$F(t, S_t) = \exp(-r(T-t)) E_Q[X|\mathfrak{I}_t] = \exp(-r(T-t)) E_Q[f(S_T)|\mathfrak{I}_t]$$

Note that because we are working risk-neutrally that $a+b = r$.

$$F(t, S_t) = \exp(-r(T)) \times$$

$$E_Q\left[f\left(S_t e^{c\left(W_T - W_t + L_{T-t} + \left(r - \frac{c^2}{2}\right)(T-t)\right)} \prod_{0 \leq s \leq T-t} (1 + \Delta L_s) e^{-\Delta L_s} \right) \right]$$

If we introduce the function,

$$F_{BS}(t, x) = \exp(-r(T-t)) E_Q\left[f\left(x \exp\left(cW_{T-t}\left(r - \frac{c^2}{2}\right)(T-t) \right) \right) \right]$$

which gives *the price of the option for the Black-Scholes model* (with volatility c), we have ,

$$F(t, x) = E_Q\left[F_{BS}\left(t, xe^{L_{T-t}} \prod_{0 < s \leq T-t} (1 + \Delta L_s) e^{-\Delta L_s} \right) \right]$$

In case of the European call for example the first two derivatives are given terms of the cumulative probability distribution function N(x) and the density function n(x) of a standart normal random variable by,

$$D_2^1 F_{BS}(t, x) = N(d_1) = N\left(\frac{\log(x/(T-t)) + (r + c^2/2)(T-t)}{c\sqrt{T-t}} \right)$$

$$D_2^2 F_{BS}(t, x) = (n(d_1)) / (xc\sqrt{T-t})$$

which are also known as the *Delta* and *Gamma* of option.

4. EQUIVALENT MARTINGALE MEASURES

In this section, we will describe the many measures, equivalent to the canonical (real world) measure under which the discounted stock price process is a martingale and under which Z remains a Lévy process. More precisely, we characterize all *Structure preserving* P-equivalent martingale measures Q under which Z remains a Levy process and the process

$$\widetilde{S} = \left\{ \widetilde{S}_t = \exp(-rt)S_t \quad , \quad t \geq 0 \right\} \quad \text{is a} \quad \{\mathfrak{I}_t\}\text{- martingale} \quad , \quad \text{where}$$

$\mathfrak{I}_t = \sigma(S_u ; 0 \leq u \leq t)$ is the naturel filtration generated by the stock price process complated with the P-null sets. Since we are considering a market with finite horizon T then $0 \leq t \leq T$ and locally equivalence for any t will be the same as equivalence.

We now want to find an equivalent martingale measure Q under which the discounted price process $\widetilde{S}$ is a martingale. Under such a Q, X has Doob-Meyer decomposition;

$$X_t = \widetilde{L}_t + \left(a + \int_{-\infty}^{+\infty} x(H(x)-1)v(dx) \right) t$$

Where, $\widetilde{L} = \left\{ \widetilde{L}_t , t \geq 0 \right\}$ is a Q- martingale. Noting that $\Delta L_t = \Delta \widetilde{L}_t$, we have,

$$\widetilde{S}_t = S_0 \exp\left(c\widetilde{W}_t + \widetilde{L}_t + \left(a + b - r + cG - \frac{c^2}{2} \right)t \right) \times$$

$$\exp\left(t \int_{-\infty}^{+\infty} x(H(x)-1)v(dx) \right) \prod_{0 < s \leq t} \left(1 + \Delta\widetilde{L}_s \right) \exp\left(-\Delta\widetilde{L}_s \right).$$

5. CONCLUSIONS

In this study, we modelling stock price process by a General Lévy Processes.Furthermore,we consider the problems of pricing contingent claims and Equivalent Martingale Measures. General Geometric Lévy market models are incomplate models and there are many equivalent martingale measures. Underlying Lévy process relating to the power-Jump processes, the market can be complated. These processes relate to the realized variation processes. Power-jump process of order two is just the variation process of degree two,i.e, the quadratic variation process and is related with the so called realized variance. Contrats on realized variance heve found their way into OTC markets and now traded regularly.

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